Please note that you have 7 pages and your mobile must be turned off and unseen

A. (70 points) Evaluate the following integrals.

1) $\int \frac{4\ln x}{x-2x\ln^2 x}dx$

$I = 4\int \frac{\ln x \quad dx}{x(1-2\ln^2 x)}$ let $u = 1-2\ln x$

$du = -\frac{2\ln x}{x}$.

$I = 2\int \frac{-du}{u} \Rightarrow -2\ln|1-2\ln^2 x| + C$

2) $\int \frac{dx}{x^2\sqrt{x^2-16}}$ $x = 4\sec\theta$.

$dx = 4\sec\theta \tan\theta \, d\theta$.

$\int \frac{4\sec\theta\tan\theta \, d\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}} = \int \frac{4\sec\theta\tan\theta \, d\theta}{16\times 4\times\sec^2\theta\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\tan\theta \, d\theta}{16\sec^2\theta|\tan\theta|}$

$0 < \theta < \frac{\pi}{2} \to \tan\theta > 0$.

$\frac{1}{16}\int \frac{1}{\sec\theta}d\theta = \frac{1}{16}\int \cos\theta = \frac{1}{16}\sin\theta$.

Now $\sec\theta = \frac{x}{4} \to \cos\theta = \frac{4}{x}$.

$I = \frac{1}{16}\sqrt{1-\frac{16}{x^2}} + C$

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3) $I = \int \frac{x-1}{x^3-x^2-42x}dx = \int \frac{x-1}{x(x+6)(x-7)} dx$;

$$= \int \frac{\cancel{x}}{\cancel{x}(x+6)(x-7)} dx - \int \frac{1}{x(x+6)(x-7)} dx$$

$$\frac{1}{(x+6)(x-7)} = \frac{A}{(x+6)} + \frac{B}{(x-7)} \qquad A \to -\frac{1}{13}, \quad B \to \frac{1}{13}.$$

$$\frac{1}{x(x+6)(x-7)} = \frac{A}{x} + \frac{B}{(x+6)} + \frac{C}{(x-7)} \qquad A \to -\frac{1}{42}, \quad B \to \frac{1}{78}, \quad C \to \frac{1}{91}$$

So, $I = -\frac{1}{13}\ln|x+6| + \frac{1}{13}\ln|x-7| + \frac{1}{42}\ln|x| - \frac{1}{78}\ln|x+6|$

$$- \frac{1}{91}\ln|x-7| + C.$$

...

4) $\int \underbrace{e^{-2x}}_{u} \underbrace{\sin 3x dx}_{dv}$

$$\left. \begin{aligned} du &= -2e^{-2x} \\ v &= \frac{-\cos 3x}{3} \end{aligned} \right\}$$

So, $I = -e^{-2x}\frac{\cos 3x}{3} - \int 2\frac{\cos^3 x}{3} e^{-2x}$

$$= -e^{-2x}\frac{\cos 3x}{3} - \frac{2}{3}\int \underbrace{\cos^3 x}_{u} \underbrace{e^{-2x}}_{dv}$$

J

$du = -2e^{-2x}$

$v = \sin 3x$.

$$J = \frac{2}{3}\left(e^{-2x}\sin 3x + 2\int e^{-2x}\sin 3x\right)$$

$$I = -e^{-2x}\frac{\cos^3 x}{3} - \frac{2}{3}e^{-2x}\sin 3x - \frac{4}{3}I.$$

$$I + \frac{4}{3}I = \frac{7}{3}I = \underline{\qquad\qquad}$$

$$I = -\frac{e^{-2x}}{7}\left(\cos^3 x + 2\sin 3x\right) + C$$

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5) $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

$$= \left[2\sqrt{|x-1|} \right]_0^2$$

$$= 2\sqrt{1} - 2\sqrt{1} = 0.$$

6) $\int \frac{dx}{2\sqrt{x}+2x}$

$$= \frac{1}{2}\int \frac{1}{x+\sqrt{x}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})}\,dx$$

let $u = \sqrt{x}+1$

$du = \frac{dx}{2\sqrt{x}}$

$$I = \frac{\cancel{2}^{\,1}}{\cancel{2}} \ln|\sqrt{x}+1| = \ln|\sqrt{x+1}|+c.$$

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7) $\int \frac{x^3 dx}{\sqrt{9+x^2}}$ let $x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$$\int \frac{27\tan^3\theta\,(3\sec^2\theta)\,d\theta}{\sqrt{9+9\tan^2\theta}} = \int \frac{27\tan^2\theta\,(3\sec^2\theta)\,d\theta}{3\,|\sec\theta|}$$

$$= \int \frac{27\tan^3\theta\,\sec^2\theta}{|\sec\theta|}\,d\theta$$ Now $\tan\theta = \frac{x}{3}$

$\Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sec\theta > 0$.

$$\Rightarrow \int 27\tan^3\theta\,\sec\theta\,d\theta$$

$$= 27\int \underbrace{\sec\theta}\ \underbrace{\tan^3\theta}\,d\theta$$

$$= 27\int \sec\theta\tan\theta\,(\tan^2\theta)\,d\theta$$

$$= 27\int \sec\theta\tan\theta\,(\sec^2\theta - 1)\,d\theta$$

$$= 27\int (\sec\theta\tan\theta\sec^2\theta - \sec\theta\tan\theta)\,d\theta$$

$$= 27\left\{\frac{\sec^3\theta}{3} - \sec\theta\right\}.$$

Now $\tan\theta \to \frac{x}{3} \to \sec^2\theta = \tan^2\theta + 1 = x^2 + 1.$

$$= 27\left[\frac{\left(\sqrt{\frac{x^2}{9}+1}\right)^3}{3} - \sqrt{\frac{x^2}{9}+1}\right] + C$$

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B. **(16 points)** Determine whether the following sequences converge or diverge. If the sequence converges, find its limit.

1) $a_n = \left(\frac{n-3}{n+1}\right)^{-n}$

$$= \left(\frac{n+1}{n-3}\right)^n$$

$$\ln a_n = n \ln \frac{n+1}{n-3} \to \infty \cdot 0.$$

$$= \frac{\ln(\;)}{\frac{1}{n}} = \frac{0}{0} \xrightarrow{\text{HoP}} \frac{\frac{\frac{+2}{(n-3)^2}}{n+1/n-3}}{+\frac{1}{n^2}} = \frac{\frac{2}{(n-3)^2} \cdot \frac{(n-3)}{(n+1)}}{\frac{1}{n^2}}$$

$$\frac{2n^2}{(n-3)(n+1)} \xrightarrow{\infty} 4.$$

$$\ln a_n = 2$$

$$e^{\ln a_n} = e^2 \Rightarrow \boxed{a_n = e^2} \therefore \text{Convergent.}$$

2) $a_n = \frac{1+\sin(\sqrt{n}\,\pi)}{\sqrt{n}}$

$$-1 \le \sin(\sqrt{n}\,\pi) \le +1$$

$$1 \le 1 + \sin(\sqrt{n}\,\pi) \le 2$$

$$\frac{1}{\sqrt{n}} \le \frac{1+\sin(\sqrt{n}\,\pi)}{\sqrt{n}} \le \frac{2}{\sqrt{n}}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0$$

By the sandwich theorem

$\Rightarrow$ an is convergent.

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C. (16 points) Answer true or false and justify.

1) $\int_{10}^{\infty} \frac{dx}{x^{3/4}+5}$ diverges

$x^{3/4} + 5 < x^{3/4}$

$\frac{1}{x^{3/4}+5} > \frac{1}{x^{3/4}}$

and $\int_{10}^{\infty} \frac{1}{x^{3/4}}$ is div (p-rule with $p = \frac{3}{4} < 1$)

then $\int_{10}^{\infty} \frac{1}{x^{3/4}+5}$ is div By DCT.

True.

2) Let f and g be two continuous functions on $[1, \infty)$ such that $0 \le f(x) \le g(x)$. If $\int_1^{\infty} f(x)\,dx$ converges, then $\int_1^{\infty} g(x)\,dx$ converges.

False.

For example. $x^2 > x$ for $1 \le x < \infty$

$\frac{1}{x^2} < \frac{1}{x}$

$\int_1^{\infty} \frac{1}{x^2}$ is conv (p rule with $p = 2 > 1$)

and $\int_1^{\infty} \frac{1}{x}$ is div (p rule with $p = 1$).

Therefore this statement is false.

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